(10.4) The cavity has $R_1 = R_2 \equiv R = 0.999$, and the finesse is therefore found from eqn 10.3 to be

$$\mathcal{F} = \frac{\pi (R^2)^{1/4}}{1-\sqrt{R^2}} = \frac{\pi\sqrt{R}}{1-R} = 3140\,.$$

Having found $\mathcal{F}$, we can use the result of Exercise 10.1 to find the cavity Q. In this exercise we have $n = 1$, $L_{\text{cav}} = 0.01\,\text{m}$, and $\lambda = 589\,\text{nm}$. The cavity mode number is then given by eqn 10.4 as:

$$m = \frac{2nL_{\text{cav}}}{\lambda} = 3.4 \times 10^4\,.$$

The cavity Q is therefore:

$$Q = m\mathcal{F} = (3.4 \times 10^4) \times (3.1 \times 10^3) = 1.1 \times 10^8\,.$$

(10.5) This exercise closely follows Example 10.1(a). Example 10.1 considers cavity 'A' from the paper by Lange *et al.*, while this Exercise considers cavity 'B' from the same reference.

(a) The condition for strong coupling is given in eqn 10.22. On inserting the relevant parameters of the cavity, we find:

$$\left(\frac{2\epsilon_0\hbar\omega V_0}{\mu_{12}^2}\right)^{1/2} = \left(\frac{2\epsilon_0\hbar(2.2\times10^{15})(5\times10^{-13})}{(3\times10^{-29})^2}\right)^{1/2} = 5\times10^7\,.$$

We thus require $Q > 5 \times 10^7$.

(b) The cavity mode number m is given from eqn 10.4 as:

$$m = \frac{2L_{\text{cav}}}{\lambda} = \frac{2\times(3.5\times10^{-4})}{8.52\times10^{-7}} = 820\,.$$

Then from the result of Exercise 10.1, we find

$$\mathcal{F} = Q/m > 5\times10^7/820 = 60\,000\,.$$

(c) The cavity finesse is related to the mirror reflectivity through eqn 10.3. In a high finesse, symmetric cavity we have $R_1 = R_2 \equiv R \approx 1$, and therefore $\mathcal{F} = \pi/(1-R)$. We thus find:

$$R = 1 - (\pi/\mathcal{F}) > 1 - (\pi/60\,000) = 0.99995\,.$$

We therefore require mirrors with reflectivities of 99.995% or higher.

(10.6) The parameters of this exercise are taken from the paper by de Martini *et al.*, *Phys. Rev. Lett.* **59**, 2955 (1987).

The scenario that we are considering is illustrated in Fig. 28. For there to be only one cavity mode within the dye spectrum, we require that one cavity mode should occur at the centre of the spectrum, and the adjacent ones should be at the edges. We thus require

$$\omega_m - \omega_{m-1} > \Delta\omega_{\text{dye}}/2\,,$$

In these circumstances, the phase change for a single pass through each layer is equal to:

$$\Delta\phi = \frac{2\pi}{\lambda} n_i d_i = \frac{2\pi}{\lambda} n_i \frac{\lambda}{4n_i} = \frac{\pi}{2}\,.$$

The phase change between light reflected from each period (i.e. the phase change between the two reflections identified in Fig. 29) is then equal to $4 \times (\pi/2) = 2\pi$, because the light must pass twice through two layers. In other words, the reflections from each period all add up in phase, so that by adding more periods, the reflectivity increases. By contrast, the phase change for the light reflected from the intermediate surface is π. The light is therefore out of phase, and adds up destructively.

(10.8) The parameters in this Exercise are taken from the paper by Gerard *et al.* [*Phys. Rev. Lett.* **81**, 1110 (1998)]. They refer to a different micropillar to the one considered in Example 10.2. We first calculate the Q of the cavity by using eqn 10.15. In a high Q cavity where $\Delta\omega$ is small, it will be the case that:

$$\frac{\omega}{\Delta\omega} \approx \frac{\lambda}{\Delta\lambda}\,,$$

so that we can find Q from:

$$Q = \frac{\lambda}{\Delta\lambda} = \frac{930}{0.18} = 5200\,.$$

The Purcell factor can then be evaluated from eqn 10.39:

$$F_{\mathrm{P}} = \frac{3Q(\lambda/n)^3}{4\pi^2 V_0} = \frac{3 \times 5200 \times (9.3 \times 10^{-7}/3.5)^3}{4\pi^2 (1.8 \times 10^{-18})} = 4.1\,.$$

(10.9) We write the equations of motion in the form:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 2\omega\Omega \\ 2\omega\Omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and look for solutions of the form:

$$\boldsymbol{r} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boldsymbol{r}_0 \mathrm{e}^{\mathrm{i}\xi t}\,.$$

On substitution this gives:

$$-\xi^2 \boldsymbol{r} = \begin{pmatrix} -\omega^2 & 2\omega\Omega \\ 2\omega\Omega & -\omega^2 \end{pmatrix} \boldsymbol{r}\,,$$

and we can therefore find ξ from

$$\begin{vmatrix} \xi^2 - \omega^2 & 2\omega\Omega \\ 2\omega\Omega & \xi^2 - \omega^2 \end{vmatrix} = 0\,.$$

This gives:

$$(\xi^2 - \omega^2)^2 - (2\omega\Omega)^2 = 0\,,$$

Chapter 12
Quantum cryptography

(12.1) The message can be decoded by subtracting the key using subtraction modulo 2:

message		1111100001101110011011000100001001101011
key	$\ominus$	1101001000110011010101100101011101000101
decoded message		0010101001011101001110100001010100101110

On dividing the decoded message into its constituent 5-bit binary letters, we find:

Binary	Decimal equivalent	Letter
00101	5	E
01001	9	I
01110	14	N
10011	19	S
10100	20	T
00101	5	E
01001	9	I
01110	14	N

The decoded message therefore reads 'EINSTEIN'.

(12.2) We follow the procedure outlined in Table 12.2. We do not need to consider the angles in this case: all we need to do is to check whether Alice's and Bob's bases are the same. If they are the same, then Bob will measure the same bit value as Alice (in the absence of an eavesdropper or errors), but if they are not, his result will be undefined.

A's data	0	0	1	0	1	1	0	0	1	0	1	1
A's basis	$\oplus$	$\oplus$	$\otimes$	$\oplus$	$\otimes$	$\otimes$	$\otimes$	$\oplus$	$\oplus$	$\otimes$	$\oplus$	$\otimes$
B's basis	$\otimes$	$\oplus$	$\oplus$	$\otimes$	$\otimes$	$\oplus$	$\oplus$	$\oplus$	$\otimes$	$\oplus$	$\otimes$	$\otimes$
Same basis ?	n	y	n	n	y	n	n	y	n	n	n	y
Sifted bits	-	0	-	-	1	-	-	0	-	-	-	1

The sifted data set shared by Alice and Bob is thus: '- 0 - - 1 - - 0 - - - 1'.

(12.3) (a) There are four possible scenarios, which are summarized in the table below.

A's data	θ_{Alice}	θ_{Bob}	θ_{det}	$\mathcal{P}_{\text{det}}$
0	0	+45	45	50%
0	0	+90	90	0
1	45	+45	90	0
1	45	+90	135	50%

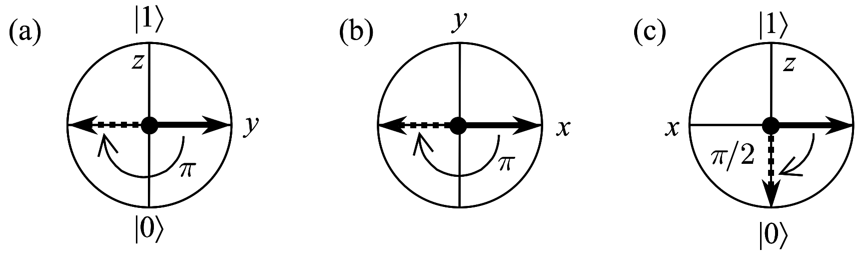

Figure 36: Bloch vector rotations, as considered in Exercise 13.4. (a) Effect of the X operator of the $(0,1,0)$ state, as observed in the y-z plane, looking down the $+x$ axis. (b)–(c) Effect of the H operator on the $(1,0,0)$ state. The operator first performs a π rotation about the z axis, as shown in part (b), and then performs a $\pi/2$ rotation about the y axis as shown in part (c).

This can be seen by direct substitution:

$$
\begin{aligned}
&\mathrm{i}\, R_z(\pi/2)\cdot R_x(\pi/2)\cdot R_z(\pi/2)\\
&\quad = \mathrm{i}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\pi/4} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\pi/4} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -\mathrm{i}/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\pi/4} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\pi/4} \end{pmatrix},\\
&\quad = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix},\\
&\quad = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix},\\
&\quad = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\\
&\quad = H\,.
\end{aligned}
$$

(13.4) (a) For $q = |1\rangle$, we have $c_0 = 0$ and $c_1 = 1$, and hence from eqn 13.17, we find:

$$
q' = Z\cdot|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle\,.
$$

In this case, the operation just produces a phase shift because the Z operator is equivalent to a π rotation about the z axis, and the $|1\rangle$ state lies along the rotation axis.

(b) For $q = (1/\sqrt{2})\,(|0\rangle + \mathrm{i}|1\rangle)$, we have $c_0 = 1/\sqrt{2}$ and $c_1 = \mathrm{i}/\sqrt{2}$. Hence from eqn 13.16 we find that:

$$
q' = X\cdot q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = (\mathrm{i}/\sqrt{2})(|0\rangle - \mathrm{i}|1\rangle)\,.
$$

By using eqn 13.7, we see that the initial and final Bloch vectors are $(0,1,0)$ and $(0,-1,0)$ respectively. The initial state lies along the $+y$ axis, and the final state along the $-y$ axis. The X operator is equivalent to a π rotation about the x axis, which would have exactly this effect, as shown in Fig. 36(a).

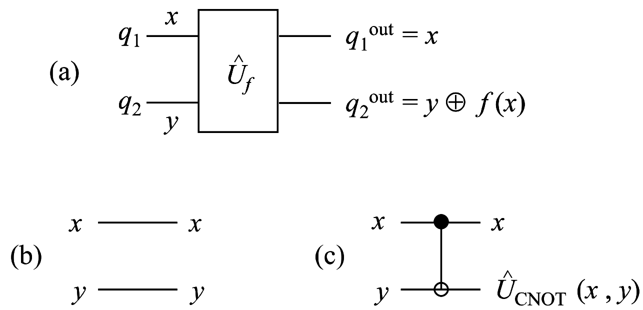

Figure 37: Operators and circuits for the Deutsch algorithm, as considered in Exercise 13.9. (a) Definition of the unitary operator. (b) Identity operation to perform the unitary operator for f_1. (c) CNOT gate to perform the unitary operation for f_3.

(c) We can use the result of part (a) to find that, for the Gaussian pulse, we have:

$$\mathcal{E}_0 = \frac{\hbar\Theta}{\sqrt{\pi}\mu_{01}\tau}\,.$$

On substituting this into the result of part (b), we find:

$$E_{\text{p}} = \frac{1}{2\sqrt{2\pi}}\,\frac{cn\epsilon_0 A\hbar^2\Theta^2}{\mu_{01}^2\tau}\,.$$

With the parameters given in the Exercise, we then find $E_{\text{p}} = 0.2\,\text{nJ}$ for $\Theta = \pi$.

(13.9) We are seeking to find the circuits for the unitary operators that perform the central part of the Deutsch algorithm for the four functions listed in Table 13.7. By comparison with Fig. 13.11, it is apparent that this unitary operator acts on two qubits q_1 and q_2, and gives an output of:

$$\begin{aligned} q_1^{\text{out}} &= q_1\,, \\ q_2^{\text{out}} &= q_2 \oplus f(x)\,, \end{aligned}$$

as shown in Fig. 37(a).

(a) When $f = f_1$, $f(x) = 0$, irrespective of the value of x. (See Table 13.7.) It will therefore be the case that

$$q_2^{\text{out}} = q_2 \oplus f(x) = q_2 \oplus 0 = q_2\,.$$

We thus have:

$$\begin{aligned} q_1^{\text{out}} &= q_1\,, \\ q_2^{\text{out}} &= q_2\,, \end{aligned}$$

which is the identity operator. The quantum circuit for the identity is trivial and is shown in Fig. 37(b).

(b) When $f(x) = f_2(x)$, we have from Table 13.7 that $f(0) = 1$ and $f(1) = 0$. The workings of the unitary operator for f_2 are then as follows:

Chapter 14
Entangled states and quantum teleportation

(14.1) We envisage an EPRB experiment as shown in Fig 14.1 in which we obtain correlated results for each event. In other words, we only ever register the results $D_1(0)D_2(0)$ or $D_1(1)D_2(1)$, and these occur with equal probability. The local hidden variables interpretation of this correlation is to postulate that the polarization of the photons is fixed as they leave the source by internal properties of the photon generation mechanism. We thus propose that the photons are generated in pairs in which the polarizations are:

- H_1H_2, leading to the result $D_1(0)D_2(0)$,
- V_1V_2, leading to the result $D_1(1)D_2(1)$.

If both pairs are generated with equal probability, then the observed results of the experiment will be reproduced.

The point about this interpretation is that the polarizations of the photons are determined before they are incident in the measuring apparatus. This interpretation is adequate for the EPRB experiment, but cannot explain the results of the Bell experiments, in which the measurement bases are different for the two photons, as in Fig. 14.8.

(14.2) The probabilities for obtaining the four possible results of the EPRB experiment for an arbitrary state $|\Psi\rangle$ are given by:

- $D_1 = 0$, $D_2 = 0$: $\mathcal{P}_{00} = |\langle 0_1, 0_2|\Psi\rangle|^2$,
- $D_1 = 0$, $D_2 = 1$: $\mathcal{P}_{01} = |\langle 0_1, 1_2|\Psi\rangle|^2$,
- $D_1 = 1$, $D_2 = 0$: $\mathcal{P}_{10} = |\langle 1_1, 0_2|\Psi\rangle|^2$,
- $D_1 = 1$, $D_2 = 1$: $\mathcal{P}_{11} = |\langle 1_1, 1_2|\Psi\rangle|^2$.

The phase factors in the wave functions are unimportant in this context.

(a) $|\langle 0_1, 0_2|\Psi\rangle|^2 = 2/3$, $|\langle 0_1, 1_2|\Psi\rangle|^2 = 0$, $|\langle 1_1, 0_2|\Psi\rangle|^2 = 0$, $|\langle 1_1, 1_2|\Psi\rangle|^2 = 1/3$. We thus observe 00 with probability 2/3 and 11 with probability 1/3.

(b) $|\langle 0_1, 0_2|\Psi\rangle|^2 = 0$, $|\langle 0_1, 1_2|\Psi\rangle|^2 = 2/5$, $|\langle 1_1, 0_2|\Psi\rangle|^2 = 3/5$, $|\langle 1_1, 1_2|\Psi\rangle|^2 = 0$. We thus observe 01 with 40% probability and 10 with probability 60%.

(c) $|\langle 0_1, 0_2|\Psi\rangle|^2 = 1/2$, $|\langle 0_1, 1_2|\Psi\rangle|^2 = 0$, $|\langle 1_1, 0_2|\Psi\rangle|^2 = 0$, $|\langle 1_1, 1_2|\Psi\rangle|^2 = 1/2$. We thus observe 00 or 11, each with 50% probability.

(14.3) The initial and final states of the atom both have $J = 0$, in which there is no angular momentum. All types of single photon transitions between $J = 0$ states are impossible, because each photon carries away at least $\hbar$

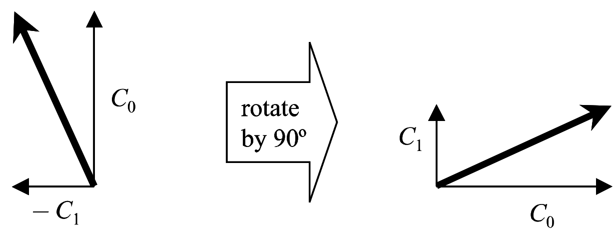

Figure 43: Effect of a clockwise 90° on the polarization state considered in Exercise 14.11.

for the case of photons for which $|0\rangle$ and $|1\rangle$ correspond to the horizontal and vertical polarization states respectively. The input and output states are illustrated in Fig. 43. It is apparent that we can transform from the initial state to the final one by applying a 90° rotation in the clockwise direction. Now a half waveplate is able to rotate a linearly-polarized photon state by an arbitrary angle by adjusting its optic axis with respect to the incoming light. We can therefore perform the unitary operation by setting the axis of the half waveplate to rotate the polarization by 90° in a clockwise direction.

(14.12) It is explained in Section 14.6 that the only case for which simultaneous events are measured on Alice's detectors D1 and D2 is for the Bell state $|\Psi^-\rangle_{12}$. It is apparent from eqn 14.23 that the four Bell-states are equally likely, and so we shall have a $|\Psi^-\rangle_{12}$ result with probability $1/4 = 25\%$. Furthermore, a Bell-state result of $|\Psi^-\rangle_{12}$ implies from eqn 14.23 that photon 3 is in the state:

$$|\psi\rangle_3 = C_0|0\rangle_3 + C_1|1\rangle_3\,,$$

that is, the same state as the input photon.

(a) When the polarizer P is set at 45°, then photon 3 will have a polarization angle of +45° when it is incident on the PBS. It will therefore go to detector D4 with 100% probability. The total probability for observing D1D2D4 events is therefore:

$$\mathcal{P} = \mathcal{P}(|\Psi^-\rangle_{12}) \times \mathcal{P}(D4) = 25\% \times 100\% = 25\%\,.$$

(b) When the polarizer P is set at 30°, the photon that arrives at the PBS has a relative polarization of −15° with respect to the +45° PBS axis. This is equivalent to a photon with polarization angle −15° being incident on a PBS set up with vertical/horizontal axes as in Fig. 12.2. The probability of an event on detector D4 is then given by eqn 12.2 as $\cos^2 15^\circ = 93\%$. The total probability for observing D1D2D4 events is therefore:

$$\mathcal{P} = \mathcal{P}(|\Psi^-\rangle_{12}) \times \mathcal{P}(D4) = 25\% \times 93\% = 23\%\,.$$

(14.13) (a) We assume that the filter gives a Lorentzian spectrum so that the coherence time is related to the spectral bandwidth by eqn 2.47. Since