Chapter 9

Resonant light-atom interactions

(9.1) We look for driven oscillations at angular frequency ω, with

$$x(t) = A\mathrm{e}^{\mathrm{i}\omega t} + B\mathrm{e}^{-\mathrm{i}\omega t}\,.$$

On substituting into the equation of motion we obtain:

$$-\omega^2\left(A\mathrm{e}^{\mathrm{i}\omega t} + B\mathrm{e}^{-\mathrm{i}\omega t}\right) + \mathrm{i}\omega\gamma\left(A\mathrm{e}^{\mathrm{i}\omega t} - B\mathrm{e}^{-\mathrm{i}\omega t}\right) + \omega_0^2\left(A\mathrm{e}^{\mathrm{i}\omega t} + B\mathrm{e}^{-\mathrm{i}\omega t}\right) = \frac{F_0}{2m_\mathrm{e}}\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right)\,.$$

On equating the terms at $+\omega$ and $-\omega$ we find:

$$\begin{aligned} A(-\omega^2 + \mathrm{i}\omega\gamma + \omega_0^2) &= \frac{F_0}{2m_\mathrm{e}}\,, \\ B(-\omega^2 - \mathrm{i}\omega\gamma + \omega_0^2) &= \frac{F_0}{2m_\mathrm{e}}\,, \end{aligned}$$

so that:

$$\begin{aligned} A &= \frac{F_0}{2m_\mathrm{e}}\frac{1}{(\omega_0^2 - \omega^2 + \mathrm{i}\omega\gamma)}\,, \\ B &= \frac{F_0}{2m_\mathrm{e}}\frac{1}{(\omega_0^2 - \omega^2 - \mathrm{i}\omega\gamma)}\,. \end{aligned}$$

It then follows that:

$$|A|^2 = |B|^2 = \left(\frac{F_0}{2m_\mathrm{e}}\right)^2 \frac{1}{[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2]}\,.$$

This describes a Lorentzian line. (cf. eqn 4.29.) The amplitude is maximum when $(\omega_0^2 - \omega^2) = 0$, i.e. $\omega = \omega_0$, as shown in Fig. 23.

The full width at half maximum is calculated by finding the values of ω such that $|A|^2$ and $|B|^2$ fall by a factor of two on either side of the maximum at $\omega = \omega_0$. This occurs when

$$(\omega_0^2 - \omega^2) \equiv (\omega_0 + \omega)(\omega_0 - \omega) = \pm\omega\gamma\,.$$

We now make use of the fact that $\omega_0 \gg \gamma$. This means that the resonance is sharply peaked at ω_0, and that the half-maximum frequencies can be written as $\omega_0 \pm \delta\omega$, where $\omega_0 \gg \delta\omega$. The half-maximum condition can then be re-written as:

$$2\omega_0|\delta\omega| = \omega_0\gamma\,,$$

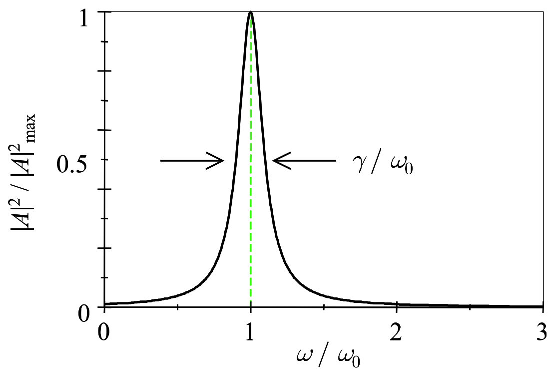

Figure 23: Square of the amplitude of the forced oscillations of a damped harmonic oscillator when driven at angular frequency ω, as considered in Exercise 9.1.

which implies:

$$|\delta\omega| = \gamma/2\,.$$

The full width at half maximum is thus (see Fig. 23):

$$\Delta\omega_{\mathrm{FWHM}} = 2\delta\omega = \gamma\,.$$

Note that we have calculated here the FWHM of the square of the amplitude rather than of the amplitude itself, i.e. of $|A|^2$ rather than $|A|$. We have done this because the power of the oscillations is proportional to $|A|^2$ rather than $|A|$. The FWHM of $|A|$ occurs when

$$\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} = \frac{1}{2\omega_0\gamma}\,,$$

which implies:

$$(\omega_0^2-\omega^2) = \pm\sqrt{3}\omega_0\gamma\,,$$

and hence an FWHM of $\sqrt{3}\gamma$.

(9.2) The density matrix for the superposition state $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$ is given by eqn 9.7.

(a) We have $c_1 = 0$ and $c_2 = 1$. Hence:

$$\boldsymbol{\rho} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) We have $c_1 = c_2 = 1/\sqrt{2}$. Hence:

$$\boldsymbol{\rho} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

(c) We have $c_1 = 1/\sqrt{3}$ and $c_2 = \mathrm{i}\sqrt{2/3}$. Hence:

$$\boldsymbol{\rho} = \begin{pmatrix} 1/3 & -\mathrm{i}\sqrt{2}/3 \\ \mathrm{i}\sqrt{2}/3 & 2/3 \end{pmatrix}.$$

(9.3) In a gas of non-degenerate two-level atoms in thermal equilibrium at temperature T, the ratio of the number of atoms in the excited state to the number in the ground state is given by Boltzmann's law:

$$N_2/N_1 = \exp(-\Delta E/k_{\mathrm{B}}T)\,,$$

where $\Delta E = E_2 - E_1$. On writing the total number of atoms as N_0, with $N_0 = (N_1 + N_2)$, we therefore have:

$$\begin{aligned}\frac{N_2}{N_0} &= \frac{\exp(-\Delta E/k_{\mathrm{B}}T)}{1+\exp(-\Delta E/k_{\mathrm{B}}T)}\,,\\ \frac{N_1}{N_0} &= \frac{1}{1+\exp(-\Delta E/k_{\mathrm{B}}T)}\,.\end{aligned}$$

Hence from eqn 9.8 we find:

$$\boldsymbol{\rho} = \frac{1}{1+\exp(-\Delta E/k_{\mathrm{B}}T)}\begin{pmatrix} 1 & 0 \\ 0 & \exp(-\Delta E/k_{\mathrm{B}}T)\end{pmatrix}\,.$$

(9.4) We follow the method of Example 9.1 and calculate the transition dipole moment μ_{12} for z-polarized light from:

$$\begin{aligned}\mu_{12} &= -e\int \psi_1^* z \psi_2 \,\mathrm{d}^3\mathbf{r}\,.\\ &= -e\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \psi_1^*\, r\cos\theta\, \psi_2\, r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\,,\end{aligned}$$

with:

$$\begin{aligned}\psi_1(r,\theta,\phi) &= \frac{1}{4\sqrt{2\pi}{a_0}^{3/2}}\left(2-\frac{r}{a_0}\right)\mathrm{e}^{-r/2a_0}\,,\\ \psi_2(r,\theta,\phi) &= \frac{\sqrt{2}}{81\sqrt{\pi}a_0^{5/2}}\left(6-\frac{r}{a_0}\right) r\cos\theta\, \mathrm{e}^{-r/3a_0}\,.\end{aligned}$$

We therefore obtain:

$$\begin{aligned}\mu_{12} &= \frac{-e}{364\pi a_0^4}\int_{\phi=0}^{2\pi}\mathrm{d}\phi\int_{\theta=0}^{\pi}\cos^2\theta\sin\theta\,\mathrm{d}\theta\\ &\quad \int_{r=0}^{\infty}\left(2-\frac{r}{a_0}\right)\mathrm{e}^{-r/2a_0}\, r\left(6-\frac{r}{a_0}\right) r\,\mathrm{e}^{-r/3a_0}\, r^2\,\mathrm{d}r\,.\end{aligned}$$

On using:

$$\begin{aligned}\int_{\phi=0}^{2\pi}\mathrm{d}\phi &= 2\pi\,,\\ \int_{\theta=0}^{\pi}\cos^2\theta\sin\theta\,\mathrm{d}\theta &= 2/3\,,\\ \int_{r=0}^{\infty}\left(12-\frac{8r}{a_0}+\frac{r^2}{a_0^2}\right)\mathrm{e}^{-5r/6a_0}\, r^4\,\mathrm{d}r &= (6/5)^6 144\, a_0^5\,,\end{aligned}$$

we find:

$$\mu_{12} = -1.769\, ea_0 = 1.497\times 10^{-29}\,\mathrm{C\,m}\,.$$

On substituting into eqn 9.43, we then obtain: $B_{12}^{\omega} = 2.39\times10^{21}\,\mathrm{m^3\,rad\,J^{-1}\,s^{-1}}$.

The A coefficient is found from eqn 4.11. The photon energy for the $n = 3 \rightarrow 2$ transition in hydrogen is given by the Bohr formula (cf eqn 3.62):

$$\hbar\omega = \left(\frac{1}{2^2} - \frac{1}{3^2}\right) R_{\mathrm{H}} = \frac{5}{36} R_{\mathrm{H}} = 1.89\,\mathrm{eV}\,.$$

In this exercise we have specified the initial and final m_l states, and hence we are considering transitions between non-degenerate sub-levels. We can therefore set $B_{12}^{\omega} = B_{21}^{\omega}$, (see eqn 4.10 with $g_1 = g_2 = 1$,) to obtain:

$$A_{21} = \frac{(\hbar\omega)^3}{\hbar^2\pi^2c^3} B_{21}^{\omega} = 2.23\times10^{7}\,\mathrm{s}^{-1}\,.$$

This can be related to the A coefficient for the $3 \rightarrow 2$ transition of hydrogen by using eqn 4.23.

(9.5) (a) We first take the time derivative of $\dot{c}_2$:

$$\ddot{c}_2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{i}}{2}\Omega_{\mathrm{R}}\,\mathrm{e}^{-\mathrm{i}\delta\omega t}c_1\right) = \frac{\mathrm{i}}{2}\Omega_{\mathrm{R}}\left(-\mathrm{i}\delta\omega\,\mathrm{e}^{-\mathrm{i}\delta\omega t}c_1 + \mathrm{e}^{-\mathrm{i}\delta\omega t}\dot{c}_1\right)\,,$$

and then substitute for c_1 and $\dot{c}_1$ from the formulae given in the exercise to obtain:

$$\begin{aligned}\ddot{c}_2 &= \frac{\mathrm{i}}{2}\Omega_{\mathrm{R}}\left(-\mathrm{i}\delta\omega\,\mathrm{e}^{-\mathrm{i}\delta\omega t}\frac{2}{\mathrm{i}\Omega_{\mathrm{R}}}\,\mathrm{e}^{+\mathrm{i}\delta\omega t}\,\dot{c}_2 + \mathrm{e}^{-\mathrm{i}\delta\omega t}\,\frac{\mathrm{i}\Omega_{\mathrm{R}}}{2}\,\mathrm{e}^{+\mathrm{i}\delta\omega t}\,c_2\right)\,,\\ &= -\mathrm{i}\delta\omega\,\dot{c}_2 - \frac{\Omega_{\mathrm{R}}^2}{4}\,c_2\,.\end{aligned}$$

Hence:

$$\ddot{c}_2 + \mathrm{i}\delta\omega\,\dot{c}_2 + \frac{\Omega_{\mathrm{R}}^2}{4}\,c_2 = 0\,.$$

(b) We use a trial solution with

$$c_2(t) = C\,\mathrm{e}^{-\mathrm{i}\zeta t}\,,$$

and substitute to obtain:

$$-\zeta^2 c_2 + \delta\omega\,\zeta c_2 + \frac{\Omega_{\mathrm{R}}^2}{4} c_2 = 0\,.$$

Hence:

$$\zeta^2 - \delta\omega\,\zeta - \frac{\Omega_{\mathrm{R}}^2}{4} = 0\,.$$

This has two roots, namely:

$$\begin{aligned}\zeta_+ &= (\delta\omega + \Omega)/2\,,\\ \zeta_- &= (\delta\omega - \Omega)/2\,,\end{aligned}$$

where $\Omega^2 = (\delta\omega)^2 + \Omega_{\mathrm{R}}^2$. Hence the general solution is:

$$c_2(t) = C_+\,\mathrm{e}^{-\mathrm{i}\zeta_+ t} + C_-\,\mathrm{e}^{-\mathrm{i}\zeta_- t}\,.$$

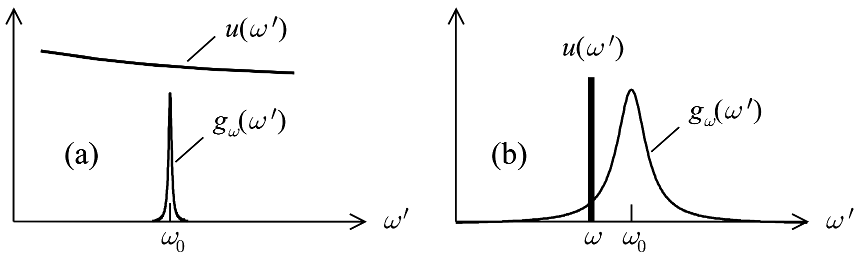

Figure 24: Two different scenarios for the interaction of radiation with an atomic transition at angular frequency ω_0, as considered in Exercise 9.6. (a) Broad-band radiation. (b) Narrow-band radiation, as from a single-mode laser. Note that the frequency scale in (b) is much expanded compared to (a), and that ω refers to the laser frequency.

(c) The initial condition that $c_2(0) = 0$ implies that:

$$C_+ + C_- = 0\,,$$

and hence that

$$c_2(t) = C_+ \left(\mathrm{e}^{-\mathrm{i}\zeta_+ t} - \mathrm{e}^{-\mathrm{i}\zeta_- t}\right)\,.$$

We substitute this into the original formula for $\dot{c}_2$ to obtain:

$$\dot{c}_2 = C_+ \left(-\mathrm{i}\zeta_+ \mathrm{e}^{-\mathrm{i}\zeta_+ t} + \mathrm{i}\zeta_- \mathrm{e}^{-\mathrm{i}\zeta_- t}\right) = \frac{\mathrm{i}\Omega_\mathrm{R}}{2}\,\mathrm{e}^{-\mathrm{i}\delta\omega t}\,c_1\,.$$

At $t = 0$ we have $c_1(0) = 1$. Therefore, on evaluating at $t = 0$, we find:

$$C_+(-\mathrm{i}\zeta_+ + \mathrm{i}\zeta_-) = \frac{\mathrm{i}\Omega_\mathrm{R}}{2}\,,$$

which implies that:

$$C_+ = -\frac{\Omega_\mathrm{R}}{2(\zeta_+ - \zeta_-)}\,.$$

Now $(\zeta_+ - \zeta_-) = \Omega$, and we thus have:

$$\begin{aligned}
c_2(t) &= -\frac{\Omega_\mathrm{R}}{2\Omega}\left(\mathrm{e}^{-\mathrm{i}(\delta\omega+\Omega)t/2} - \mathrm{e}^{-\mathrm{i}(\delta\omega-\Omega)t/2}\right)\,,\\
&= -\frac{\Omega_\mathrm{R}}{2\Omega}\,\mathrm{e}^{-\mathrm{i}\delta\omega t/2}\left(\mathrm{e}^{-\mathrm{i}\Omega t/2} - \mathrm{e}^{+\mathrm{i}\Omega t/2}\right)\,,\\
&= \frac{\mathrm{i}\Omega_\mathrm{R}}{\Omega}\,\mathrm{e}^{-\mathrm{i}\delta\omega t/2}\,\sin(\Omega t/2)\,.
\end{aligned}$$

Hence:

$$|c_2(t)|^2 = \frac{\Omega_\mathrm{R}^2}{\Omega^2}\,\sin^2(\Omega t/2)\,.$$

(9.6) The two scenarios that we are considering are illustrated schematically in Fig. 24. In the first case considered in part (a) of the Exercise, it is valid to consider the radiation energy density to be constant over the spectral width of the transition, as shown in Fig. 24(a). In the second

case considered in parts (b)–(d), the radiation band-width is much smaller than the atomic line width, as illustrated in Fig. 24(b). This second case applies, for example, when a narrow band laser interacts with an atomic transition.

(a) With a broad-band, slowly-varying source we can treat $u(\omega')$ as a constant over the spectral line shape. On making use of the definition of the spectral line shape function given in eqn 4.26, we then find that the total upward stimulated transition rate is given by:

$$\begin{aligned} W_{12}^{\text{total}} &= \int_0^\infty W_{12}(\omega')\,\mathrm{d}\omega'\,, \\ &= \int_0^\infty N_1 B_{12}^{\omega} u(\omega') g_\omega(\omega')\,\mathrm{d}\omega'\,, \\ &= N_1 B_{12}^{\omega} u(\omega_0) \int_0^\infty g_\omega(\omega')\,\mathrm{d}\omega'\,, \\ &= N_1 B_{12}^{\omega} u(\omega_0)\,, \end{aligned}$$

which, on re-writing ω_0 as ω, gives the same transition rate as in eqn 4.5. Similarly, the total downward stimulated transition rate is given by:

$$\begin{aligned} W_{21}^{\text{total}} &= \int_0^\infty W_{21}(\omega')\,\mathrm{d}\omega'\,, \\ &= \int_0^\infty N_1 B_{21}^{\omega} u(\omega') g_\omega(\omega')\,\mathrm{d}\omega'\,, \\ &= N_1 B_{21}^{\omega} u(\omega_0) \int_0^\infty g_\omega(\omega')\,\mathrm{d}\omega'\,, \\ &= N_1 B_{21}^{\omega} u(\omega_0)\,, \end{aligned}$$

as in eqn 4.6.

(b) We substitute for $u(\omega')$ and again integrate over ω' to find the total upward stimulated transition rate:

$$\begin{aligned} W_{12}^{\text{total}} &= \int_0^\infty W_{12}(\omega')\,\mathrm{d}\omega'\,, \\ &= \int_0^\infty N_1 B_{12}^{\omega} u_\omega \delta(\omega' - \omega) g_\omega(\omega')\,\mathrm{d}\omega'\,. \end{aligned}$$

On making use of the identity:

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x)\,\mathrm{d}x = f(a)\,,$$

we then find:

$$W_{12}^{\text{total}} = N_1 B_{12}^{\omega} u_\omega g_\omega(\omega)\,.$$

Similarly, the downward stimulated transition rate is given by:

$$\begin{aligned} W_{21}^{\text{total}} &= \int_0^\infty W_{21}(\omega')\,\mathrm{d}\omega'\,, \\ &= \int_0^\infty N_2 B_{21}^{\omega} u_\omega \delta(\omega' - \omega) g_\omega(\omega')\,\mathrm{d}\omega'\,, \\ &= N_2 B_{21}^{\omega} u_\omega g_\omega(\omega)\,. \end{aligned}$$

(c) The time dependence of the population of the upper level is given by:

$$\begin{aligned}\frac{\mathrm{d}N_2}{\mathrm{d}t} &= W_{12} - W_{21} - A_{21}N_2\,,\\ &= N_1 B_{12}^{\omega} u_\omega g(\omega) - N_2 B_{21}^{\omega} u_\omega g(\omega) - A_{21}N_2\,.\end{aligned}$$

In an ideal two-level system, the total number of atoms must satisfy the constraint:

$$N_0 = N_1 + N_2 = \text{constant}\,.$$

We can then substitute for N_1 to obtain, after dividing through by N_0:

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_2/N_0) = (1-N_2/N_0)B_{12}^{\omega}u_\omega g(\omega) - (N_2/N_0)B_{21}^{\omega}u_\omega g(\omega) - A_{21}(N_2/N_0)\,.$$

On writing $N_2/N_0 = x$ and $B' = B_{12}^{\omega} g_\omega(\omega) = B_{21}^{\omega} g_\omega(\omega)$, we then have:

$$\frac{\mathrm{d}x}{\mathrm{d}t} + (2B'u_\omega + A_{21})x = B'u_\omega\,.$$

We multiple through by the integrating factor e^{Ct} to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x\mathrm{e}^{Ct}) = B'u_\omega \mathrm{e}^{Ct}\,,$$

where $C = 2B'u_\omega + A_{21}$. On integrating we then find:

$$x\mathrm{e}^{Ct} = (B'u_\omega/C)\mathrm{e}^{Ct} + \text{constant}\,.$$

The boundary condition $N_2(0) = 0$ implies $x(0) = 0$, and hence that the constant is equal to $-B'u_\omega/C$. The solution is therefore:

$$\frac{N_2}{N_0} \equiv x(t) = \frac{B'g_\omega}{2B'u_\omega + A_{21}}\left(1 - \mathrm{e}^{-(2B'u_\omega + A_{21})t}\right)\,.$$

(d) (1) For very intense fields we have a very large value of u_ω, and hence $B'u_\omega \gg A_{21}$. We can therefore ignore the factor of A_{21} in the denominator of the pre-factor, making it apparent that:

$$\lim_{t\to\infty}(N_2/N_0) = 1/2\,,$$

and hence that the populations tend to equalize. This result can be understood by realizing that for intense fields we can ignore spontaneous emission. At $t = 0$ all the atoms are in the ground state and there will be net absorption upwards to the excited state. As the population of the upper level increases, the stimulated emission rate increases. Eventually, when $N_2 = N_1 = N_0/2$, the stimulated absorption and emission rates are identical, and a dynamic balance is achieved leading to no further change in the populations.

(2) In the weak field limit we put $B'u_\omega \ll A_{21}$, so that

$$\lim_{t\to\infty}(N_2/N_0) = B'u_\omega/A_{21} \equiv (B_{12}^{\omega}/A_{21})\, g_\omega(\omega)\, u_\omega\,.$$

The population of the upper level is therefore proportional to the light intensity, and also to the atomic line shape, as we would expect.

(9.7) The fluorescence reaches its first maximum when the pulse area defined in eqn 9.51 is equal to π.

(a) For a 'top hat' pulse, the electric field is constant for a time T and zero at all other times. The integral of the electric field over time is therefore equal to $\mathcal{E}_0 T$, and we thus have:

$$\frac{\mu_{12}\mathcal{E}_0 T}{\hbar} = \pi\,,$$

which implies:

$$\mathcal{E}_0 = \pi\hbar/\mu_{12}T = 3.3 \times 10^3\,\mathrm{V/m}\,.$$

The intensity is then found from eqn 2.28 to be equal to $1.4 \times 10^4\ \mathrm{W\,m^{-2}}$.

(b) It is shown in Example 9.3 that the pulse area obtained for a Gaussian pulse is equal to:

$$\Theta = \sqrt{\pi}\mu_{12}\mathcal{E}_{\mathrm{peak}}\tau/\hbar\,,$$

where $\tau = \tau_{\mathrm{FWHM}}/1.177$. On setting the pulse area equal to π, we then find:

$$\mathcal{E}_{\mathrm{peak}} = 1.177\sqrt{\pi}\hbar/\mu_{12}\tau_{\mathrm{FWHM}} = 2.2 \times 10^3\,\mathrm{V/m}\,.$$

The peak intensity is then found from eqn 2.28 to be equal to $6.2 \times 10^3\,\mathrm{W\,m^{-2}}$.

(9.8) The experimental data tells us that $\Omega_{\mathrm{R}} = 2\pi \cdot 78 \times 10^6$ rad/s. We are told in the caption that the optical intensity was $6400\,\mathrm{W\,m^{-2}}$, and hence from eqn 2.28 we deduce that the electric field amplitude was 2200 V/m. We can then deduce the dipole moment from eqn 9.32:

$$\mu_{12} = \frac{\hbar\Omega_{\mathrm{R}}}{\mathcal{E}_0} = \frac{4.9 \times 10^8\,\hbar}{2200} = 2.4 \times 10^{-29}\,\mathrm{C\,m}\,.$$

(9.9) With c_1 and c_2 as defined in eqn 9.64, we find:

$$\begin{aligned} c_1c_2 &= \mathrm{e}^{\mathrm{i}\varphi}\,\sin(\theta/2)\,\cos(\theta/2)\,,\\ &= \mathrm{e}^{\mathrm{i}\varphi}\,\sin\theta/2\,,\\ &= (\cos\varphi + \mathrm{i}\sin\varphi)\sin\theta/2\,. \end{aligned}$$

Hence:

$$\begin{aligned} \mathrm{Re}\langle c_1c_2\rangle &= \cos\varphi\,\sin\theta/2\,,\\ \mathrm{Im}\langle c_1c_2\rangle &= \sin\varphi\,\sin\theta/2\,, \end{aligned}$$

which is consistent with eqn 9.63 with x and y defined in eqn 9.61 and $r = 1$. Similarly, we find:

$$\begin{aligned} |c_2|^2 - |c_1|^2 &= \cos^2(\theta/2) - \sin^2(\theta/2)\,,\\ &= \cos\theta\,, \end{aligned}$$

which is again consistent with eqn 9.63 for z with $r = 1$.

(9.10) The coefficients of the superposition states can be related to the angles of the Bloch vector by using eqn 9.64.

(a) For $\theta = \pi/2$ and $\varphi = 0$, we find $c_1 = c_2 = 1/\sqrt{2}$. Hence:

$$\psi = (1/\sqrt{2})(|1\rangle + |2\rangle)\,.$$

(b) For $\theta = \pi/2$ and $\varphi = \pi/2$, we find $c_1 = 1/\sqrt{2}$ and $c_2 = \mathrm{i}/\sqrt{2}$. Hence:

$$\psi = (1/\sqrt{2})(|1\rangle + \mathrm{i}|2\rangle)\,.$$

(c) For $\theta = \pi/2$ and $\varphi = \pi$, we find $c_1 = 1/\sqrt{2}$ and $c_2 = -1/\sqrt{2}$. Hence:

$$\psi = (1/\sqrt{2})(|1\rangle - |2\rangle)\,.$$

(d) For $\theta = \pi/2$ and $\varphi = -\pi/2$, we find $c_1 = 1/\sqrt{2}$ and $c_2 = -\mathrm{i}/\sqrt{2}$. Hence:

$$\psi = (1/\sqrt{2})(|1\rangle - \mathrm{i}|2\rangle)\,.$$

(e) For $\theta = \pi/3$ and $\varphi = \pi/4$, we find $c_1 = 1/2$ and $c_2 = \sqrt{3/8}(1+\mathrm{i})$. Hence:

$$\psi = (1/2)|1\rangle + \sqrt{3/8}(1+\mathrm{i})|2\rangle\,.$$

(9.11) We can relate the wave function to the position on the Bloch sphere by using eqn 9.63.

(a) For $c_1 = 1/\sqrt{3}$ and $c_2 = \sqrt{2/3}$, we have

$$\begin{aligned} c_1 c_2 &= \sqrt{2}/3\,, \\ |c_2|^2 - |c_1|^2 &= 1/3\,. \end{aligned}$$

Hence:

$$(x, y, z) = (\sqrt{8}/3, 0, 1/3)\,.$$

(b) For $c_1 = \sqrt{2/3}$ and $c_2 = -\mathrm{i}/\sqrt{3}$, we have

$$\begin{aligned} c_1 c_2 &= -\mathrm{i}\sqrt{2}/3\,, \\ |c_2|^2 - |c_1|^2 &= -1/3\,. \end{aligned}$$

Hence:

$$(x, y, z) = (0, -\sqrt{8}/3, -1/3)\,.$$

(c) For $c_1 = \mathrm{e}^{\mathrm{i}\pi/4}/\sqrt{2}$ and $c_2 = 1/\sqrt{2}$, we have

$$\begin{aligned} c_1 c_2 &= \mathrm{e}^{\mathrm{i}\pi/4}/2\,, \\ |c_2|^2 - |c_1|^2 &= 0\,. \end{aligned}$$

Hence:

$$(x, y, z)(1/\sqrt{2}, 1/\sqrt{2}, 0)\,.$$

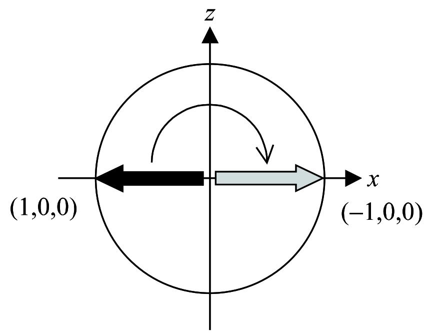

Figure 25: Effect of a π-pulse on the state $(|1\rangle + |2\rangle)/\sqrt{2}$ when the rotation is carried out about the y axis, as considered in Exercise 9.13.

(9.12) In a statistical mixture, the atoms are either in level $|1\rangle$ with $c_1 = 1$ and $c_2 = 0$ or in level $|2\rangle$ with $c_1 = 0$ and $c_2 = 1$. The value of $\langle c_1 c_2 \rangle$ is therefore equal to zero for every atom. On the other hand, the expectation values of $|c_2|^2$ and $|c_1|^2$ are respectively equal to 60% and 40%. We thus have from eqn 9.63:

$$\begin{aligned} x &= 2\mathrm{Re}\langle c_1 c_2 \rangle = 0\,, \\ y &= 2\mathrm{Im}\langle c_1 c_2 \rangle = 0\,, \\ z &= |c_2|^2 - |c_1|^2 = 0.2\,. \end{aligned}$$

The statistical mixture thus corresponds to the point $(0, 0, 0.2)$. Note that this point is not on the surface of the sphere. In a statistical mixture, all phase information has been lost due to scattering events, which have the effect of causing the Bloch vector to move towards the z axis as shown in Fig. 9.10(a).

(9.13) This exercise closely follows Example 9.3.

(a) The pulse area is given by:

$$\Theta = \sqrt{\pi}\mu_{12}\mathcal{E}_{\mathrm{peak}}\tau/\hbar\,,$$

where $\tau = \tau_{\mathrm{FWHM}}/1.177 = 2.55\,\mathrm{ps}$. Hence the peak electric field in the π-pulse where $\Theta = \pi$ is given by:

$$\mathcal{E}_{\mathrm{peak}} = \hbar\sqrt{\pi}/\mu_{12}\tau = 9.2 \times 10^5\,\mathrm{V/m}\,.$$

The calculation of the pulse energy proceeds exactly as in Example 9.3, except that we have the extra factor of n in the intensity (see eqn 2.28). We thus have:

$$E_{\mathrm{pulse}} = \sqrt{\frac{\pi}{8}} A n c \epsilon_0 \mathcal{E}_{\mathrm{peak}}^2 \tau\,.$$

Hence for $A = \pi(2 \times 10^{-6})^2\,\mathrm{m}^2$ and $n = 3.5$, we find $E_{\mathrm{pulse}} = 160\,\mathrm{fJ}$.

(b) We first calculate the initial state on the Bloch sphere by using eqn 9.63. With $c_1 = c_1 = 1/\sqrt{2}$, we find $\langle c_1 c_2 \rangle = 1/2$ and $|c_2|^2 - |c_1|^2 = 0$, giving:

$$(x, y, z)_{\mathrm{initial}} = (1, 0, 0)\,.$$

A π-pulse will produce a rotation of the Bloch vector by $180°$ about its appropriate axis. In this case, we are rotating about the y axis, and so the final position on the Bloch sphere is equal to $(-1, 0, 0)$, as illustrated in Fig. 25. This Bloch vector has $\theta = \pi/2$ and $\varphi = \pi$, and hence from eqn 9.64 we find $c_1 = 1/\sqrt{2}$ and $c_1 = -1/\sqrt{2}$. The final state is therefore:

$$\psi_{\text{final}} = (1/\sqrt{2})(|1\rangle - |2\rangle)\,.$$

(9.14) For each pulse the Bloch vector is rotated by an angle Θ_i equal to

$$\Theta_i = \frac{\mu_{12}}{\hbar}\int_{-\infty}^{+\infty} \mathcal{E}_i(t)\,\mathrm{d}t = \frac{\mu_{12}\mathcal{E}_i\tau_i}{\hbar}\,.$$

The axis of the rotation is determined by the phase of the pulse. The initial state is at the South pole of the Bloch sphere, and we set up the axes so that the initial pulse rotates the Bloch vector in the x-z plane.

(a) The relative phases of the two pulses are the same, and so both rotations are about the y axis. The rotation angles are given by:

$$\begin{aligned}
\Theta_1 &= \frac{2\times10^{-29}\cdot 4139\cdot 1\times10^{-9}}{\hbar} = \frac{\pi}{4}\,,\\
\Theta_2 &= \frac{2\times10^{-29}\cdot 6209\cdot 2\times10^{-9}}{\hbar} = \frac{3\pi}{4}\,.
\end{aligned}$$

The total rotation angle about the y axis is thus π. This moves the atom to the North pole of the Bloch sphere, as illustrated in Fig. 26 (a). The final state is thus $|2\rangle$.

(b) The rotation angles for the two pulses are given by:

$$\begin{aligned}
\Theta_1 &= \frac{2\times10^{-29}\cdot 827.8\cdot 1\times10^{-8}}{\hbar} = \frac{\pi}{2}\,,\\
\Theta_2 &= \frac{2\times10^{-29}\cdot 3311\cdot 5\times10^{-9}}{\hbar} = \pi\,.
\end{aligned}$$

The first pulse rotates about the y axis, while the second pulse rotates about the axis in the x-y plane at $90°$ with respect to the y axis, i.e. the x axis. The effects of the two rotations are illustrated in Fig. 26(b). The first pulse moves the Bloch vector up to the x axis, and then the second pulse has no effect. We therefore end up with the Bloch vector with $\theta = \pi/2$ and $\varphi = 0$ in polar co-ordinates, which gives from eqn 9.64: $c_1 = c_2 = 1/\sqrt{2}$. The final state is thus:

$$\psi = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\,.$$

(c) The rotation angles for the two pulses are given by:

$$\begin{aligned}
\Theta_1 &= \frac{2\times10^{-29}\cdot 2.758\times10^{4}\cdot 3\times10^{-10}}{\hbar} = \frac{\pi}{2}\,,\\
\Theta_2 &= \frac{2\times10^{-29}\cdot 5.519\times 104\cdot 3\times10^{-10}}{\hbar} = \pi\,.
\end{aligned}$$

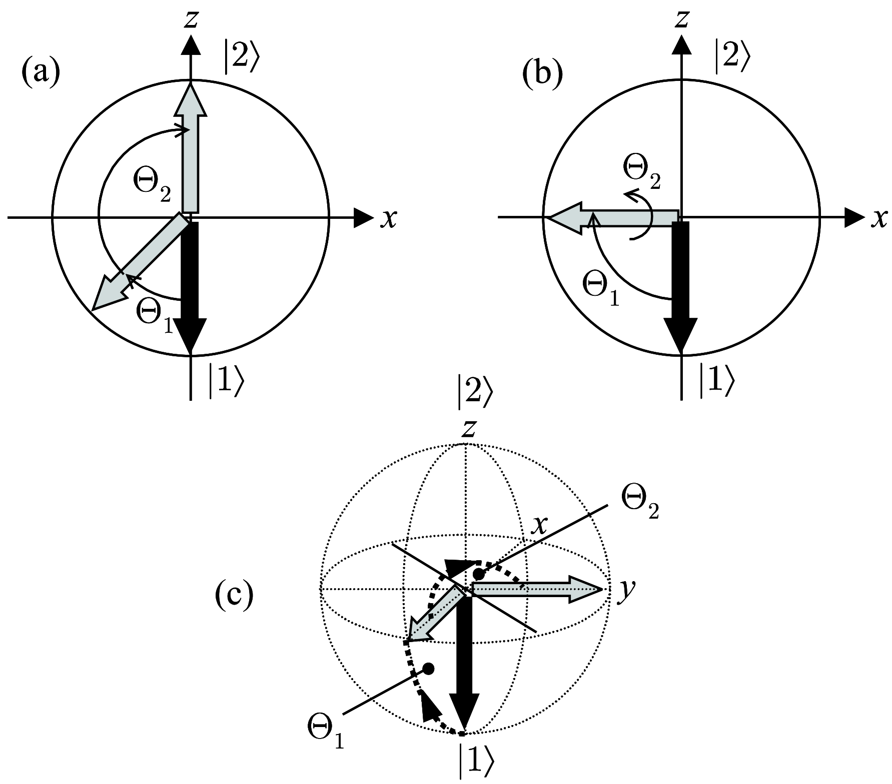

Figure 26: Rotations of the Bloch vector by two pulses, as considered in Exercise 9.14. (a) $\pi/4$ rotation about y axis followed by a $3\pi/4$ rotation about the same axis. (b) $\pi/2$ rotation about y axis followed by a π rotation about the x axis. (c) $\pi/2$ rotation about y axis followed by a π rotation about an axis in x-y plane at 45° to the y axis.

The first pulse rotates about the y axis, while the second pulse rotates about the axis in the x-y plane at 45° with respect to the y axis. The effects of these two rotations are illustrated in Fig. 26(c). The first pulse moves the Bloch vector up to the x axis, and then the second pulse moves the Bloch vector onto the y axis. We therefore end up with the Bloch vector with $\theta = \pi/2$ and $\varphi = \pi/2$ in polar co-ordinates, which gives from eqn 9.64: $c_1 = 1/\sqrt{2}$ and $c_2 = -\mathrm{i}/\sqrt{2}$. The final state is thus:

$$\psi = \frac{1}{\sqrt{2}}(|1\rangle - \mathrm{i}|2\rangle)\,.$$