Vacuum isn't so simple:

Quantum Fields, Coherent States, and Squeezed States

Phys 552/ECE 695 Quantum Optics and Photonics
Lecture 14: Quantization of Light, Coherent and Squeezed States
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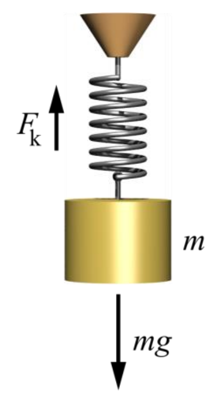
As we have discussed in previous lectures, we can use photon statistics (i.e. Super-Poissonian, Poissonian, or Sub-Poissonian) along with correlation functions to determine the type of light source we are observing. However, we also know that light can be viewed as either a stream of photons or as a wave of electric and magnetic fields. Understanding quantized light using the photon picture is relatively straightforward and can be experimentally verified using the aforementioned statistics, but it is not as easily understood when considering the wave-nature of light. This consideration led to the birth of quantum electrodynamics (QED), which is a very active field of research today. In this short review, we will cover the idea of quantized electromagnetic fields and draw comparisons to the quantum harmonic oscillator model. Using this analogy, we can uncover very interesting physics related to zero-point vacuum energy, spontaneous emission, and quantum uncertainty. In addition to derivations and discussion, which can be found in more detail in *Quantum Optics: An Introduction* by Fox (FQ) Ch. 7, I'll provide several references for papers related to the study of the topics we will cover.

1. Light as a classical harmonic oscillator

We begin by recalling the basic formulas associated with the traditional harmonic oscillator model, namely the position and momentum in time. For the mass on a spring model (shown to the right), the equations are as follows:



$$x(t) = x_o \sin(\omega t) \tag{1.1}$$

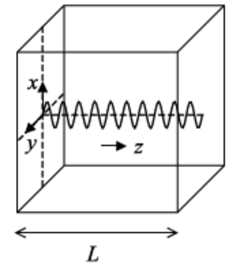
$$p(t) = p_o \cos(\omega t) \tag{1.2}$$

$$\omega = \sqrt{\frac{k}{m}} \quad \text{and} \quad p_o = m\omega x_o$$

and energy of the harmonic oscillator is given as:

$$u_{HO} = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

One important take-away from this is to realize that the position and momentum are 90° out of phase. Therefore, to enable the analogy between the harmonic oscillator and fields, the $\vec{E}$ and $\vec{B}$ should also be 90° out of phase. In general this is not true for propagating waves but is true for standing waves (i.e. in a cavity). We recall from standard electromagnetic analysis with appropriate boundary conditions that an $\hat{x}$ polarized $\vec{E}$ and $\vec{B}$ are given as:



$$\vec{E}_x(z,t) = E_o \sin(kz)\sin(\omega t) \tag{1.3}$$

$$\vec{B}_y(z,t) = \frac{E_o}{c}\cos(kz)\cos(\omega t) \tag{1.4}$$

The modes allowed in this cavity are determined by $k = \frac{2\pi}{\lambda} = n\frac{\pi}{L}$, so that the fields are quantized in the cavity. The energy in the cavity is the volume integral of the field energy density:

$$u_{cavity} = \int_V U dV = \int_V \frac{1}{2}\left(\varepsilon_o \left|\vec{E}\right|^2 + \frac{1}{\mu_o}\left|\vec{B}\right|^2\right) dV \tag{1.5}$$

$$u_{cavity} = \frac{V}{4}\left(\varepsilon_o E_o^2 \sin^2(\omega t) + \frac{B_o^2}{\mu_o}\cos^2(\omega t)\right) \tag{1.6}$$

By observing the terms in the energy of the cavity and the energy of the harmonic oscillator, we can complete the analogy by making the following definitions:

$$q(t) = \sqrt{\frac{\varepsilon_o V}{2\omega^2}} E_o \sin(\omega t) \tag{1.7}$$

$$p(t) = \sqrt{\frac{V}{2\mu_o}} B_o \cos(\omega t) \tag{1.8}$$

such that the energy is written as:

$$u = \frac{1}{2}(p^2 + \omega^2 q^2) \tag{1.9}$$

where q and p are related to the original position and momentum of the harmonic oscillator as:

$$q(t) = \sqrt{m}\, x(t) \tag{1.10}$$

$$p(t) = \frac{1}{\sqrt{m}} p_x(t) \tag{1.11}$$

Which mirror the harmonic oscillator equations given in (1.1) and (1.2). We can also see that $\vec{E}$ can be viewed as proportional to the potential energy and $\vec{B}$ proportional to the kinetic energy. One might argue that this doesn't apply for most situations since the wave should be in a cavity to obtain quantization. However, one could consider that the limits of the universe can impose boundary conditions (i.e. edge of space-time as it expands from the big-bang), thereby leading to quantized electromagnetic fields in all space (i.e. everything is inside a cavity). Of course, such an understanding of the universe is highly debatable, and borders religion and philosophy as much as science, but the point is that there is a potential explanation for why such behavior could be physical.

2. Phasor diagrams and field quadratures

We recall that complex electromagnetic fields can be represented on a complex plane with an amplitude and an angle in the form $A_o e^{i\phi}$. The projections onto the real and imaginary axis are then given as $A = A_o \cos\phi + iA_o \sin\phi = A_1 + iA_2$. It is traditional in quantum optics to have the field

amplitudes be dimensionless so we multiply the real and imaginary field amplitudes (A_1 and A_2 respectively) by a factor $E_o\sqrt{\varepsilon_o V / 4\hbar\omega}$. The resulting amplitudes are called field quadratures given as:

$$X_1(t) = \sqrt{\varepsilon_o V / 4\hbar\omega}\, E_o \sin\omega t = \sqrt{\omega / 2\hbar}\, q(t) \tag{1.12}$$

$$X_2(t) = \sqrt{\varepsilon_o V / 4\hbar\omega}\, E_o \cos\omega t = \sqrt{1 / 2\hbar\omega}\, p(t) \tag{1.13}$$

3. *Light as a quantum harmonic oscillator*

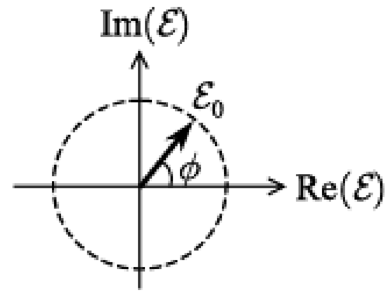
With the analogy between electromagnetic waves in a cavity and a harmonic oscillator established, we can then draw conclusions on quantized electromagnetic waves based on the quantum harmonic oscillator. Namely, the energy of a level n is given by:

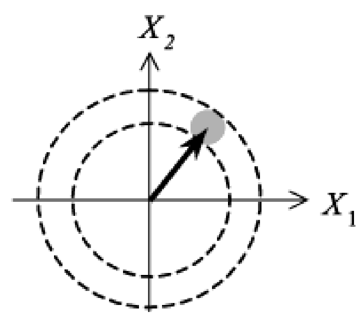
$$u_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{1.14}$$

where n represents the number of photons and the Heisenberg uncertainty principle

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} \tag{1.15}$$

An important consequence of this analysis is that with n=0 (i.e. no photons in the cavity) there is a non-zero energy given by $u_0 = \frac{\hbar\omega}{2}$. This mysterious energy is called the zero-point energy and we will come back to it in a following section for a more in depth discussion including current research. Using the relations between the position and momentum of the harmonic oscillator and the field quadratures X_1 and X_2, given through (1.10 – 1.13), we can find the uncertainty of the electromagnetic harmonic oscillator. Thus, we see that quantized light has uncertainty in its field quadratures, which can be schematically represented by the shaded circle in the phasor diagram to the right.





$$\Delta X_1 \Delta X_2 = \sqrt{\frac{\omega}{2\hbar}}\Delta q \sqrt{\frac{1}{2\hbar\omega}}\Delta p = \frac{1}{2\hbar}\left(\frac{\Delta x}{\sqrt{m}}\right)\left(\sqrt{m}\Delta p_x\right) = \frac{1}{2\hbar}\Delta x \Delta p_x \tag{1.16}$$

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4} \tag{1.17}$$

4. *Zero-point energy and the vacuum field*

As we saw in the previous section, likening electromagnetic fields to a quantum harmonic oscillator has an important consequence, the zero-point energy. This energy exists even when no photons (n=0) are present in the system, but how is this possible? Mathematically, it arrives from the uncertainty

principle, but is physically understood through the constant creation and annihilation of particle and anti-particle pairs (i.e. virtual particles) in the vacuum, resulting in energy fluctuations and a lowest possible energy that is non-zero [1]. As Christopher Ray said in his book, “It is a mistake to think of any physical vacuum as some absolutely empty void”. To understand this effect and its consequences further, let us consider the field associated with the zero-point energy. If we assume that the electric and magnetic fields contain the same energy we can write the total energy as:

$$2\int_V \frac{1}{2}\varepsilon_o \left|\vec{E}_{vac}\right|^2 dV = \frac{1}{2}\hbar\omega \tag{1.18}$$

which leads to a value of the vacuum field of:

$$E_{vac} = \sqrt{\frac{\hbar\omega}{2\varepsilon_o V}} \tag{1.19}$$

It is important to note that the field is inversely proportional to the volume of the cavity or space of interest. For most cases, this field is negligible, but for nanotechnology applications it may be appreciable (i.e. for V = 1 μm^3 E_{vac} = 1.5x10^5 V/m). In fact, the effects of this field have been experimentally observed and measured through the Casimir force. The Casimir force is an attractive force between two objects when placed very close together and results from the vacuum field and fluctuations. This force is proportional to the separation distance to the fourth power ($F \propto L^{-4}$) and is quite small, but has been experimentally confirmed. Also, in certain instances the Casimir force can be repulsive, resulting in potential applications to levitation [2].

Another important consequence of the vacuum field relates to spontaneous emission. Spontaneous emission has always been considered a natural consequence of an emitter in an excited state and an irreversible process [3]. Despite this, it is well known that the effect can still be controlled to some extent, as suggested by Purcell, by controlling the surrounding environment. However, when the quantum vacuum picture is invoked, spontaneous emission can be considered as stimulated emission resulting from vacuum fluctuations. In this sense, it can be engineered and even greatly suppressed [4-6].

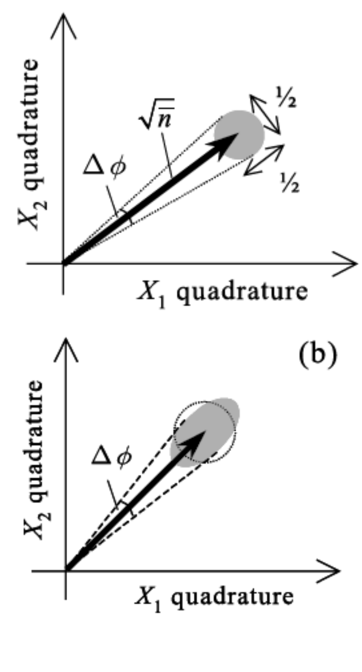
5. Coherent and squeezed states

In the previous sections, the uncertainty in the quantized electromagnetic field is given by Eq 1.17. This equation can also be likened to the phasor form:

$$A = X_1 + iX_2 = \left|\alpha\right| e^{i\phi} \tag{1.20}$$

$$\left|\alpha\right| = \sqrt{X_1^2 + X_2^2} \text{ and } \phi = \tan^{-1}\frac{X_2}{X_1}$$

Here, were can consider several important situations in the uncertainties of the field quadratures. First, when $\Delta X_1 = \Delta X_2 = \frac{1}{2}$. This is called a coherent state and is the quantum equivalent of a monochromatic



electromagnetic wave obeying Poisson statistics so that the average. Thus, the uncertainty region in the phasor diagram (shaded region) would be a perfect circle. However, we can have unequal uncertainties which produce a squeezed state in either the amplitude term of the phasor or in the phase term of the phasor. These squeezed states of light are very important for obtaining extremely high sensitivities. One such example is in the light interferometer gravitational wave (LIGO) experiment which is attempting to observe gravitiy waves using a Michelson interfereometer. The gravity waves will distort space-time causing the distance between the mirrors and the beam splitter to be altered and interference to be detected. However, extreme phase sensitivity must be obtained ~10^{-18}. By using squeezed light with a large uncertainty in the amplitude, this level of phase accuracy can be reached.

Squeezed light can be generated through second order nonlinear processes. In this case, a laser at frequency 2ω sent into a nonlinear crystal which a second order nonlinear response with a second signal beam at ω. Through the nonlinear interaction, an idler beam is produced which has a frequency of 2ω - ω = ω. This idler is degenerate with the signal and allows for parametric amplification or de-amplification based on the phase difference between the signal and the pump. If we consider that vacuum fluctuations provide the signal input at frequency ω, we can see amplification and de-amplification of the vacuum field based on the phase difference. Because the amplitude of the idler is smaller than the base vacuum for some phases, resulting in quadrature-squeezed vacuum states. These, along with other techniques have been used to generate squeezed states [7-9].

The same techniques can be applied to other means of amplification. When amplifying a signal, say with a standard gain medium, the quantum noise is also amplified. However, by using phase sensitive amplification like the nonlinear process mentioned above, one of the phase quadratures can be amplified with nearly no noise, at the expense of increased noise in the other.

[1] APS, “The Force of Empty Space,” *Physical Review Focus*, vol. 2, 1998.
[2] I. Dzyaloshinskii et al "The general theory of van der Waals forces". *Advances in Physics,* vol. 10, no. 38, 1961.
[3] H. Yokoyama and K. Ujihara, *Spontaneous Emission and Laser Oscillation in Microcavities,* CRC Press, 1995.
[4] D. Kleppner “Inhibited spontaneous emission,” Physical Review Letters, vol. 47, no. 4, 1981.
[5] P. Lodahl et al. “Controlling the dynamics of spontaneous emission from quantum dots by photonic crystals,” *Nature*, vol. 430, 2004.
[6] H. Lee et al. “Quenching of spontaneous emission via quantum interference,” *Physical Review A*, vol. 55, no. 6, 1997.
[7] L. Wu et al “Generation of squeezed states by parametric down conversion”, *Physical Review Letters*, vol. 57 ,no. 20, 1986.
[8] P. Lam et al “Optimization and transfer of vacuum squeezing from an optical parametric oscillator,” *Journal of Optics B: Quantum and Semiclassical Optics*, vol. 1, no. 4, 1999.
[9] R. Slusher et al “Pulse squeezed light”, *Physical Review Letters*, vol. 59, no. , 1987.
[10] M. Fox, *Quantum Optics: An Introduction*, Oxford University Press, 2006.