The Bloch Sphere: Developing a graphical way of viewing the two-level system

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1 Atom Light Interaction

The interaction between a two-level system, specifically an atom interacting with light with frequency at or near the transition frequency between two of the atomic energy levels, is a rich area of study within quantum optics. Today's lecture covers the mathematical analysis of the two level system and how Rabi oscillations come about and also introduces a very useful graphical method for analyzing the system's state, known as the Bloch sphere.

1.1 Bloch Sphere

The Bloch sphere attaches a geometric meaning to probabilities and phases, and so it gives a very useful way of "seeing" how various parameters change. It is typically used for two level systems because, as we'll see later, these systems can be parametrized as points on the surface of a unit sphere. The method here can be generalized to systems with higher level states, but it is obviously best for a two-level system since that yields a three-dimensional Bloch sphere. A Bloch "sphere" in higher dimensions is no longer helpful as a visualization tool!

1.1.1 Preliminary definitions for the two-level system

Throughout this document, we will denote the two basis vectors of our two-level system as

$$\begin{aligned} |g\rangle &= \text{ground state} \\ |e\rangle &= \text{excited state} \end{aligned} \tag{1}$$

Notice that for many physical systems the two-level state is an approximation. For example, atomic systems have many energy levels, and yet they can be treated as two-level systems so long as the light that they interact with is near the transition frequency of two specific states; transitions to other state will then be off-resonance and will be smaller order effects. If we desire, these can be treated by the incorporation of a damping term (see chapter 9 for more information on this). Other systems, like spins in a magnetic field, can truly be treated as two level systems. This analysis can be easily generalized to any two level system, which makes it a very powerful idea.

Upon making the assumption that our system has these two basis vectors, we can write any state as a linear combination in this basis:

$$|\psi\rangle = c_g(t)|g\rangle + c_e(t)|e\rangle. \tag{2}$$

The key insight underlying the Bloch sphere is that with the proper mapping function this last relationship defines a **unit sphere**. One way to identify this is to make the mapping

$$\begin{aligned} x &= 2\mathrm{Re}\left[C_g \cdot C_e^*\right] \\ y &= 2\mathrm{Im}\left[C_g \cdot C_e^*\right] \\ z &= |c_e|^2 - |c_g|^2. \end{aligned} \tag{3}$$

Since $|c_g(t)|^2 + |c_e(t)|^2 = 1$, x, y, and z as defined above also are normalized to be a unit sphere:

$$x^2 + y^2 + z^2 = 1. \tag{4}$$

Since cartesian coordinates are an awkward choice for an inherently spherical system, let's define an alternative mapping that makes it clearer that this defines the surface of a unit sphere. We can write

$$C_g = \sin(\theta/2), \quad C_e = e^{-i\phi}\cos(\theta/2), \tag{5}$$

where θ, ϕ are the same angular variables as we use in spherical coordinates. If you're concerned that these two mappings might not be consistent, take a look at exercise 9.9. We can demonstrate the physical information contained within this mapping by computing the probabilities that the system is in either state from $P_e = |C_e|^2$ and $P_g = |C_g|^2$. Doing so, we see that

$$P_e = \frac{1}{2}(1 + \cos\theta), \;\; P_g = \frac{1}{2}(1 - \cos\theta). \tag{6}$$

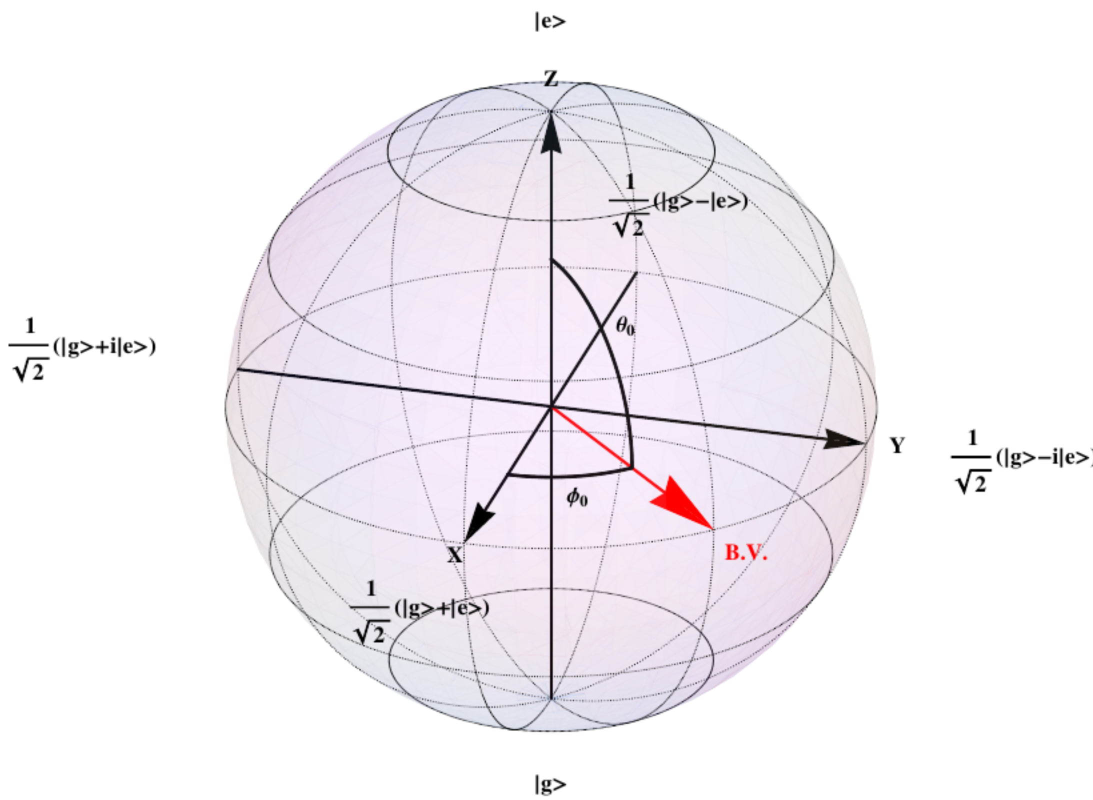
As all good probabilities should, these sum to unity. Additionally, they show that θ is physically meaningful - it's what determines the relative probabilities! Notice that ϕ gives the relative phase between these two components, and thus will give interference effects. Let's continue this exploration by seeing the effect θ has on the different probabilities. We quickly see that:

- $\theta = 0 \implies P_e = 1$; thus, the north pole corresponds to $|\psi\rangle = |e\rangle$.
- $\theta = \pi \implies P_g = 1$; thus, the south pole corresponds to $|\psi\rangle = |g\rangle$.
- anywhere in the xy plane, we have an equal superposition.
- Points on opposite sides of the sphere are orthogonal.

Some important conclusions that we'll derive during the mathematical analysis are that

- If we illuminate the atoms with a resonant laser, we travel around the sphere at a constant latitude.
- The time evolution of atoms in a superposition state corresponds to traveling around the equator.

Now that we've talked about the Bloch sphere a lot, let's see what it looks like!



We can see how the axes are defined by different superposition states.

1.2 Two Level System - Nearly Resonant Light

To see how the Bloch sphere helps us visualize the relevant parameters in a two-level state problem, let's analyze this scenario by solving the relevant mathematics. We need to make some assumptions about this problem:

1. We approach this semi-classically - so the atoms behave quantum mechanically, but the light is treated as a classical wave. No photons!

2. We also assume that the electric field of the light does not change much over the atomic scale: this is called the **dipole approximation**. Assuming this we write the potential term for the Hamiltonian as

$$\hat{V} = e\vec{r} \cdot \vec{E}. \tag{7}$$

3. Assume that light is close enough to resonance that the 2-level physics dominate anything else - this is what I've mentioned earlier.

With these assumptions in place, we can go ahead and do the quantum mechanics. This is an exactly solvable problem now - a rarity in real physics! The following is more of a synopsis of results as done by Mr. Blasing rather than thorough calculations; for all the steps take a look at chapter 9 of the Fox Optics book. The approach in that text differs in some regards from what is done here, but most the main results and steps are the same.

Let's start with a generic electric field

$$\vec{E} = \vec{E}_0 \cos(\omega t + \delta). \tag{8}$$

We can write our Hamiltonian H as the sum of two components: the free evolution piece,

$$\hat{H}_{FE} = \hbar\omega_0 |e\rangle\langle e|, \tag{9}$$

which gives the free energy of the two states (note the ground state is set to zero energy), and the atom-field interaction piece

$$\hat{H}_{AF} = \frac{\hbar\Omega}{2}\left[e^{i(\omega t+\delta)}|g\rangle\langle e| + e^{-i(\omega t+\delta)}|e\rangle\langle g|\right], \tag{10}$$

where

$$\Omega = \frac{\vec{\mu}_{eg}\cdot\vec{E}_0}{\hbar} \tag{11}$$

is the so-called Rabi frequency and $\vec{\mu}_{eg}$ is the matrix element that determines how well $\vec{r}$ couples the excited state to the ground state. It is defined as

$$\vec{\mu}_{eg} = -e\langle e|\vec{r}|g\rangle. \tag{12}$$

We're now ready to solve the time-dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle, \tag{13}$$

where ψ is given by equation (2). Doing so yields the coupled differential equations

$$\begin{aligned} \dot{C}_g(t) &= -i\frac{\Omega}{2}e^{i(\omega t+\delta)}C_e(t) \\ \dot{C}_e(t) &= -i\omega_0 C_e(t) - i\frac{\Omega}{2}e^{-i(\omega t+\delta)}C_g(t), \end{aligned} \tag{14}$$

where ω_0 is called the intrinsic frequency (it's the transition frequency between our two states), and is defined

$$\omega_0 = \frac{(E_e - E_g)}{\hbar}. \tag{15}$$

Now in principle we can go ahead and solve these two coupled differential equations. The math will be simpler, however, if we can go to the right "frame" - one that "rotates" with our system. The approach Mr. Blasing used in the lecture was to apply a unitary transformation to the system. This helps since the differential equations written above have a high-frequency piece - $e^{-i\omega t}$ - which can be thought of as "rotating" us around the Bloch sphere. If we rotate along with us, the dynamics become clearer - that's the

1.2.1 Frame Transformation

Unitary transformations express a basis change, so by doing this to our system we basically move between bases in Hilbert-space. Consider the unitary transformation

$$U = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega t} \end{bmatrix}, \tag{16}$$

such that $UU^\dagger = U^\dagger U = 1$. U is labeled the basis transformation, while $U^\dagger$ is called the coefficient transformation. When we apply this to our basis states we get

$$|\widetilde{g}\rangle = U|g\rangle = |g\rangle,\ |\widetilde{e}\rangle = U|e\rangle = e^{i\omega t}|e\rangle. \tag{17}$$

These are our new basis vectors in the "rotating" frame. Likewise, the coefficients transform to

$$\widetilde{C}_g = C_g,\ \widetilde{C}_e = e^{-i\omega t} C_e. \tag{18}$$

It's important to note that our wavefunction ψ is independent of the coordinate system used - we're used to that with vectors, and this is the exact same sort of idea. So even though our coefficients and basis states have been changed, the wavefunction is exactly the same, which is exactly what we'd expect after acting on it with a unitary transformation. Alright, so after doing these transformations, we obtain the new coupled differential equations:

$$\begin{aligned} \dot{\widetilde{C}}_g(t) &= -i\frac{\Omega}{2} e^{i\delta} \widetilde{C}_g e(t) \\ \dot{\widetilde{C}}_e(t) &= i\Delta \widetilde{C}_e(t) - i\frac{\Omega}{2} e^{-i\delta} \widetilde{C}_g(t), \end{aligned} \tag{19}$$

where we've defined another quantity $\Delta = \omega - \omega_0$, which is called the "detuning" - it gives how far the laser frequency is from the intrinsic frequency of the system. Obviously if we have a perfectly resonant laser, $\Delta = 0$ and these equations are simpler to solve! Let's do that case first:

1.3 General Solution

1.3.1 No Detuning

If we assume no detuning, then the general solution to these two equations is

$$\begin{aligned} \widetilde{C}_g(t) &= \widetilde{A}_g \cos\left(\frac{\Omega_{gen} t}{2}\right) + \widetilde{B}_g \sin\left(\frac{\Omega_{gen} t}{2}\right) \\ \widetilde{C}_e(t) &= \widetilde{A}_e \cos\left(\frac{\Omega_{gen} t}{2}\right) + \widetilde{B}_e \sin\left(\frac{\Omega_{gen} t}{2}\right), \end{aligned} \tag{20}$$

here we've defined $\Omega_{gen} = \sqrt{\Omega^2 + \Delta^2}$ and called it the generalized Rabi frequency.

Actually, there was a little sleight of hand that's been done in obtaining these equations. Throughout terms that oscillate at a frequency $\omega + \omega_0$ have been discarded. The reasoning for this is two-fold: first, since these are such fast oscillations, they will average out to zero over the typical experimental time span. Additionally, any terms that oscillate at this factor are also divided by it, while terms that depend on $\omega - \omega_0$ will be divided by that factor. For on-resonance laser forcing, the latter denominator will be very small, while $\omega + \omega_0$ will be large. Eliminating these terms is therefore equivalent to our assumption that this is a perfect two state system - we're ignoring all non-resonant terms.

1.3.2 With Detuning

If we take detuning into account, all we actually need to do is multiply both coefficients by $e^{i\Delta t/2}$. The constants A and B are rather complicated:

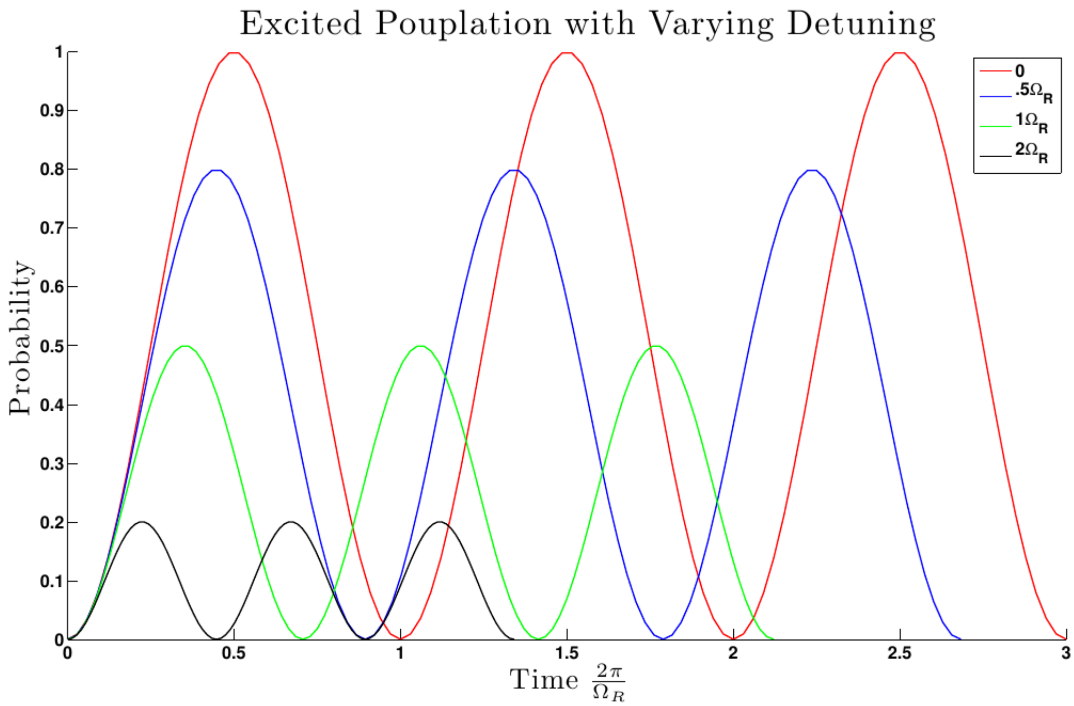
$$\begin{aligned} \widetilde{A}_g &= \sin\frac{\theta_2}{2} \\ \widetilde{B}_g &= -\frac{i\Delta}{\Omega_{gen}} \sin\frac{\theta_0}{2} - \frac{i\Omega}{\Omega_{gen}} \cos\frac{\theta_0}{2} e^{i(\delta - \phi_0)} \\ \widetilde{A}_e &= \cos\frac{\theta_2}{2} e^{-i\phi} \\ \widetilde{B}_e &= \frac{i\Delta}{\Omega_{gen}} \cos\frac{\theta_0}{2} e^{-i\phi} - \frac{i\Omega}{\Omega_{gen}} \sin\frac{\theta_0}{2} e^{-i\delta}. \end{aligned} \tag{21}$$

1.4 Analysis

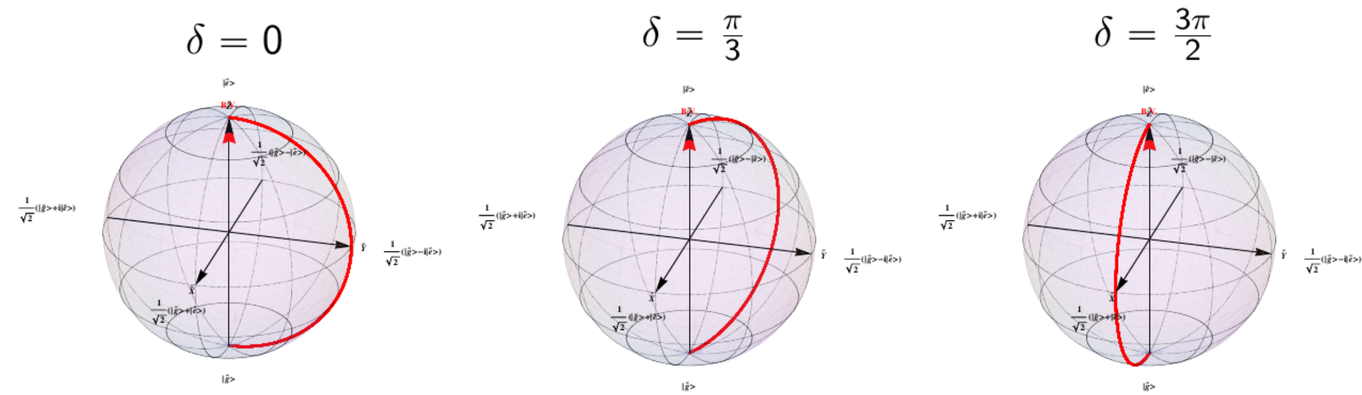
Let's start off in the ground state: $\phi = 0$, $\theta = \pi$. Then,

$$P_e = \left(\frac{\Omega}{\Omega_{gen}}\right)^2 \sin\left(\Omega_{gen} t/2\right)^2 . \tag{22}$$

This is just what we've seen in chapter 9 - Rabi oscillations! Notice how the amount of detuning, which is reflected in the difference between Ω and Ω_{gen}, affects whether or not we get total population inversion or not. We can plot the probability for different detuning amount and see this:



It's also interesting to see the effect of the laser's phase: it determines the plane in which the oscillations take place:



2 Conclusion

In conclusion, we've worked through most of the two-level system problem and obtained some of the key results. We can see how the Bloch sphere helps us analyze the behavior of this system since we can match points on its surface with various superposition states, and track Rabi oscillations by rotating a vector around the Bloch sphere.